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COMMENT

Some general properties of master equations

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Abstract. Haken's three assertions on master equations, normalisation of transition probability, existence and uniqueness of a stationary distribution, and approach to the stationary distribution, are not always true when the system has infinitely many states. In particular, the stochastic Lotka–Volterra model does not necessarily conclude the approach to a state in which both preys and predators are extinct.

Recently Haken (1978) discussed some aspects of the stochastic Lotka–Volterra model

$$\begin{aligned}\partial_t P(t,(x,y)) = {} & A(x-1)P(t,(x-1,y)) - AxP(t,(x,y)) \\ & +(x+1)(y-1)P(t,(x+1,y-1)) - xyP(t,(x,y)) \\ & +B(y+1)P(t,(x,y+1)) \\ & -ByP(t,(x,y)) \qquad x,y=0,1,\ldots \qquad A,B>0 \end{aligned} \tag{1}$$

based on general properties of master equations.

He asserted that a solution $P(t, x)$ of a master equation on a state space S

$$\partial_t P(t,x) = \sum_{z\neq x} w(z,x)P(t,z) - \sum_{z\neq x} w(x,z)P(t,x) \qquad x,z\in S \tag{2}$$

($w(x, y)$ represents probability per unit time to jump from x to y, which is written as $w(y, x)$ in Haken's notation) satisfies the following.

(i) If $0 \leq P(0,x) \leq 1$, $\forall x \in S$, and $\Sigma_{x\in S} P(0,x) = 1$, then

$$0 \leq P(t,x) \leq 1 \qquad \forall x \in S \tag{3}$$

$$\sum_{x\in S} P(t,x) = 1 \qquad \text{for any } t>0. \tag{4}$$

(ii) There exists at least one stationary solution $P_{st}(x)$ such that $\partial_t P_{st}(x) = 0$ and $\Sigma_{x\in S} P_{st}(x) = 1$.

(iii) If the stationary solution is unique, then $P(t, x) \to P_{st}(x)$ as $t \to \infty$.

(iv) Suppose any two points $x, y \in S$ are connected by some sequence $x_1, x_2, \ldots, x_n$ such that

$$w(x_i, x_{i-1}) > 0 \qquad \text{or} \qquad w(x_{i-1}, x_i) > 0 \tag{5}$$

for $i = 1, 2, \ldots, n+1$ ($x_0 = x$, $x_{n+1} = y$). Then the stationary solution is unique.

Haken concluded by using (iv) that (1) has only one stationary solution $P_{st}(x, y) = \delta_{x,0}\delta_{y,0}$, which had been proved by Reddy (1973) with a long calculation. On the basis of (iii) he claimed $P(t, x, y) \to \delta_{x,0}\delta_{y,0}$ as $t \to \infty$.

Great care must be taken, however, when we apply (i)–(iv) to a process with an infinite state space S as is the case of the Lotka–Volterra model. We present in this note some examples violating (i)–(iv). Schnackenberg (1976) asserted (i)–(iv), imposing an additional condition

$$w(x, y)>0 \Rightarrow w(y, x)>0 \tag{6}$$

which the Lotka–Volterra model fails to satisfy; but properties (i) and (ii) are not valid in general.

Let us consider a master equation of birth–death type on a state space $S=\{0, 1, \ldots\}$:

$$\partial_t P(t, 0) = -\lambda_0 P(t, 0) + \mu_1 P(t, 1) \tag{7}$$

$$\partial_t P(t, x) = -(\lambda_x + \mu_x) P(t, x) + \lambda_{x-1} P(t, x-1) + \mu_{x+1} P(t, x+1) \qquad x = 1, 2, \ldots$$

with an initial condition $P(0, x) = \delta_{x, x_0}$.

We first show that the master equation (7) with $x_0 = 0$ does not have a solution having the properties (3) and (4) when

$$\lambda_x = \lambda^x \qquad \mu_x = \mu^x \qquad \lambda > 1, 0 < \mu \leqslant 1. \tag{8}$$

It is worth noting that this model fulfils condition (6). Suppose there exists a solution $P(t, x)$ of (7) satisfying (3) and (4). By virtue of (7), $F(t, x) := \Sigma_{z=0}^{x} P(t, z)$ satisfies

$$\partial_t F(t, x) = -\lambda_x P(t, x) + \mu_{x+1} P(t, x+1).$$

Integration with respect to t yields

$$1 - F(t, x) = \lambda_x \int_0^t P(s, x)\, \mathrm{d}s - \mu_{x+1} \int_0^t P(s, x+1)\, \mathrm{d}s.$$

Assumption (3) assures $1 - F(t, x) \leqslant 1$, hence

$$\lambda_x \int_0^t P(s, x)\, \mathrm{d}s - \mu_{x+1} \int_0^t P(s, x+1)\, \mathrm{d}s \leqslant 1.$$

Using this inequality recursively, we have

$$\begin{aligned} \lambda_x \int_0^t P(s, x)\, \mathrm{d}s &\leqslant 1 + \sum_{l=1}^{L-1} \prod_{z=x+1}^{x+l} \frac{\mu_z}{\lambda_z} + \mu_{x+L} \prod_{z=x+1}^{x+L-1} \frac{\mu_z}{\lambda_z} \int_0^t P(s, x+L)\, \mathrm{d}s \\ &\leqslant 1 + \sum_{l=1}^{\infty} (\mu/\lambda)^{x+l} + \int_0^t P(s, x+L)\, \mathrm{d}s. \end{aligned} \tag{9}$$

Here we have used $\mu \leqslant 1, \lambda > 1$. By assumptions (3) and (4) $\lim_{L\to\infty} P(t, x+L) = 0$ and $P(s, x+L) \leqslant 1$, so that we have

$$\lim_{L\to\infty} \int_0^t P(s, x+L)\, \mathrm{d}s = 0 \tag{10}$$

by the bounded convergence theorem. From (9) and (10)

$$\lambda_x \int_0^t P(s, x)\, \mathrm{d}s \leqslant 1 + (\mu/\lambda)^{x+1} (1 - \mu/\lambda)^{-1}.$$

Summing this with respect to x, we obtain

$$\int_0^t F(s,x)\,ds \leqslant \sum_{x=0}^{\infty} \lambda^{-x} + \lambda(1-\mu/\lambda)^{-1} \sum_{x=0}^{\infty} (\mu/\lambda^2)^{x+1}.$$

Noting $F(s,x)\uparrow 1$ as $x\to\infty$, and again invoking the bounded convergence theorem, we have

$$t \leqslant (1-1/\lambda)^{-1} + \mu\lambda^{-1}(1-\mu/\lambda)^{-1}(1-\mu/\lambda^2)^{-1}$$

which is a contradiction for sufficiently large t.

We can always construct a solution $\bar{P}$ of (7), sometimes called a minimal solution, with a property (3) and a substochastic property

$$\sum_{x=0}^{\infty} \bar{P}(t,x) \leqslant 1 \tag{11}$$

for any $t>0$. To define $\bar{P}$, consider a backward equation of (7) in an integral form

$$P(t,x,y) = \exp[-(\lambda_x+\mu_x)t]\delta_{x,y} + \int_0^t \exp[-(\lambda_x+\mu_x)(t-s)][\lambda_x P(s,x+1,y) + \mu_x P(s,x-1,y)]. \tag{12}$$

Here we have explicitly written the starting point x, i.e. $P(0,x,y)=\delta_{x,y}$. The minimal solution $\bar{P}(t,x,y)$ is defined as a limit of a sequence P_n approximating (12):

$$P_0(t,x,y) = \delta_{x,y}\exp[-(\lambda_x+\mu_x)t]$$

$$P_n(t,x,y) = P_0(t,x,y) + \int_0^t \exp[-(\lambda_x+\mu_x)(t-s)] \times[\lambda_x P_{n-1}(s,x+1,y) + \mu_x P_{n-1}(s,x-1,y)] \qquad n\geqslant 1. \tag{13}$$

It is not difficult to show by induction that P_n is an increasing function of n and satisfies (3) and (11); hence $\bar{P} = \lim_{n\to\infty} P_n$ exists and also has the properties (3) and (11). $\bar{P}$ is shown to be a solution of the forward equation (7) (Feller 1966).

Secondly, we show that the master equation (7) with (8) does not have a stationary solution; if it does, say P_{st}, we have from (7)

$$-\lambda_x P_{st}(x) + \mu_{x+1}P_{st}(x+1) = 0 \qquad x=0,1,2,\ldots.$$

Then P_{st} is given by

$$P_{st}(x) = \prod_{l=1}^{x} (\lambda_{l-1}/\mu_l)P_{st}(0) \qquad x\geqslant 1$$

which fails to fulfil the normalisation condition $\Sigma_{x\in S} P_{st}(x)=1$ since

$$A := 1 + \sum_{k=1}^{\infty}\prod_{l=1}^{k} (\lambda_{l-1}/\mu_l) = \infty. \tag{14}$$

We must note that a stationary solution does not always exist even if (4) holds. As an example still satisfying (6) we consider

$$\lambda_x = 1 \qquad \mu_{x+1}=1 \qquad x=0,1,2,\ldots. \tag{15}$$

This model clearly has no stationary solution since A defined by (14) is infinite. On the other hand, its minimal solution $\bar{P}$ fulfils

$$\begin{aligned}0 &\leqslant 1-\sum_{l=0}^{x} \bar{P}(t, l) \\ &\leqslant \int_0^t \bar{P}(s, x)\,\mathrm{d}s-\int_0^t \bar{P}(s, x+1)\,\mathrm{d}s \\ &\leqslant \int_0^t \bar{P}(s, x)\,\mathrm{d}s. \end{aligned} \tag{16}$$

Since $\bar{P}$ has the properties (3) and (11), $\lim_{x\to\infty} \int_0^t \bar{P}(s, x)\,\mathrm{d}s = 0$, which implies (4).

Thirdly, we give an example that $P(t, x)$ does not approach the stationary solution $P_{st}(x)$ even if it exists uniquely. The simplest one may be

$$\lambda_0 = 0 \qquad \mu_1 = 1 \qquad \lambda_i = 1 \qquad \mu_{i+1} = 0 \qquad i = 1, 2, \ldots \tag{17}$$

and $P(0, x) = \delta_{x,1}$. This model is a pure birth process having a trap at $x = 0$. The equation (7) reads

$$\begin{aligned}\partial_t P(t, 0) &= P(t, 1) \\ \partial_t P(t, 1) &= -2P(t, 1) \\ \partial_t P(t, x) &= -P(t, x)+P(t, x-1) \qquad x = 2, 3, \ldots\end{aligned} \tag{18}$$

and is solved as

$$\begin{aligned}P(t, 0) &= \tfrac{1}{2}(1-\mathrm{e}^{-2t}) \\ P(t, 1) &= \mathrm{e}^{-2t}\end{aligned}$$

which shows that $P(t, 0)$ tends to $\frac{1}{2}$ as $t \to \infty$.

Let us give one more example with $P(0, x) = \delta_{x,1}$ and

$$\lambda_x = 1-(1+x)^{-2} \qquad \mu_{x+1} = (2+x)^{-2} \qquad x = 0, 1, \ldots$$

which has the property (6) on $\{1, 2, \ldots\}$. It can be shown that $\Sigma_{x=0}^{\infty} P(t, x) = 1$, $\forall t > 0$ in almost the same way as was done for (15). As is well known (Feller 1966) a sample starting from x stays there for time τ_x with $E[\tau_x] = 1$, and then hits $x+1$ $(x-1)$ with probability $1-(1+x)^{-2}$ $((1+x)^{-2})$. Consider an event M for which a transition $1 \to 2 \to 3 \to \ldots$ occurs. Then the fact $\Sigma_{x=0}^{\infty} P(t, x) = 1$ means $\Sigma_{x=1}^{\infty} \tau_x = \infty$, and

$$P(M) = \prod_{n=2}^{\infty} (1-n^{-2}) = \tfrac{1}{2}$$

i.e. Prob (samples starting from $\{1\}$ do not hit $\{0\}$) $\geqslant \frac{1}{2}$.

The stochastic Lotka–Volterra model is another example violating (iii). As is clear from figure 1, every sample starting from (x, y), $x \geqslant 1$ hits the set $(z, 0)$ $(z = 1, 2, 3, \ldots)$ with strictly positive probability, and after reaching it the sample never goes to $(0, 0)$.

As for the property (iv), it is not true even when S is a finite set. Such a case is given by: $S = \{1, 2, 3, 4\}$, $w(1, 4) = w(3, 2) = 1$, $w(1, 2) = w(3, 4) = 2$ and all the other $w(x, y) = 0$. This has infinitely many stationary solutions $P_{st}(2) = \alpha$, $P_{st}(4) = 1-\alpha$,

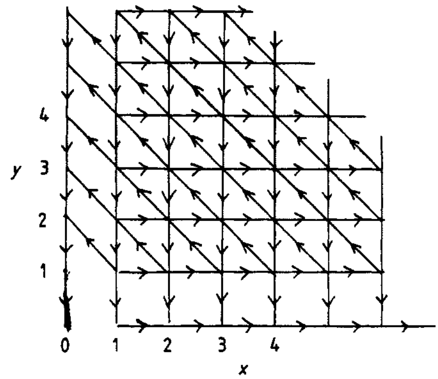

Figure 1. Digraph for the stochastic Lotka–Volterra model. An arrow is drawn from x to y if $w(x, y) > 0$.

$P_{st}(1) = P_{st}(3) = 0$, $\alpha \in [0, 1]$. The property (v) is found to be valid, which is obtained from (iv) by replacing (5) by

$$\prod_{i=0}^{n} w(x_i, x_{i+1}) > 0.$$

Proof of (v) and sufficient conditions for (i)–(iii) are given by developing the Lyapunov method. All we have to do is to find a Lyapunov function of the backward operator of (2). This will be discussed later (Ito 1981).

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